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# STABILITY OF A FAST MAGNETOHYDRODYNAMIC SHOCK WAVE IN PLASMA WITH ANISOTROPIC PRESSURE

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Various models of magnetic hydrodynamics (including the so-called Chew–Goldberger–Low model [1, 2]) are widely used for describing real processes in some fields of physics and engineering such as astrophysics, high-velocity aerodynamics, etc. In these processes, as is known, the medium often moves with strong discontinuities, for instance, shock waves. Therefore, the problem on the stability of strong discontinuities (including shock waves) generates interest in magnetic hydrodynamics with anisotropic pressure.

The stability of strong discontinuity has not yet been studied comprehensively, even in the usual magnetic hydrodynamics, in contrast to, say, gas dynamics [3, 4]. Indeed, after the publication of classical works [5, 6], only a few subsequent papers can be noted (see, e.g., [7, 8]) in which the stability of strong discontinuity is discussed to some extent.

To study the problem on stability of fast shock waves in magnetic hydrodynamics with anisotropic pressure, we have used an equational approach which implies the study of the well-posedness of the linear mixed problem on stability of fast shock waves. The main point of the study is the construction of an a priori estimate indicating the correctness of this problem. The most complete description of the approach in application to the problems of the usual magnetic hydrodynamics is presented in [9]. Two particular cases of the above linear mixed problem on stability of a fast shock wave in magnetic hydrodynamics with anisotropic pressure were considered in [10], and the well-posedness of the problem was proved by constructing an a priori estimate using the technique of energy dissipation integrals.

Dealing with the general statement of the linear mixed problem on the stability of a fast magnetohydrodynamic shock wave in plasma with anisotropic pressure, the present paper proves the well-posedness of the mixed problem, thus corroborating the stability of this type of strong discontinuities under some assumptions on the parameters characterizing the initial stationary discontinuity.

**1. Equations of Anisotropic Magnetic Hydrodynamics and Strong Discontinuity Relations.** In the Chew–Goldberger–Low approximation, the motion of collisionless plasma in the high magnetic field is described by the following system of equations (see [1, 2, 11]):

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (\rho v_i)_t + \sum_{k=1}^{3} (\Pi_{ik})_{x_k} = 0, \quad i = 1,2,3, \tag{1.1}$$

$$\mathbf{H}_t - \operatorname{rot}(\mathbf{v}\times\mathbf{H}) = 0, \quad (\rho S^{\|})_t + \operatorname{div}(\rho \mathbf{v} S^{\|}) = 0, \quad (\rho S^{\perp})_t + \operatorname{div}(\rho \mathbf{v} S^{\perp}) = 0.$$

Here, $\rho$ is the plasma density, $\mathbf{v} = (v_1, v_2, v_3)^*$ is the plasma velocity, $t$ is time, and $\mathbf{x} = (x_1, x_2, x_3)$ is the vector in Cartesian coordinates, $\Pi_{ik} = \rho v_i v_k + \mathcal{P} b_i b_k + \mathcal{P}^{\perp}\delta_{ik}$; $\Pi_{ik} = \rho v_i v_k + \mathcal{P} b_i b_k + \mathcal{P}^{\perp}\delta_{ik}$ are the components of the momentum flux tensor; $\mathcal{P} = p^{\|} - p^{\perp} - w^2/(4\pi)$, $\mathcal{P}^{\perp} = p^{\perp} + w^2/(8\pi)$, where $p^{\|}$ and $p^{\perp}$ are the longitudinal and transverse pressures respectively; $w = |\mathbf{H}|$; $\mathbf{b} = (b_1, b_2, b_3)^* = \mathbf{H}/w$, where $\mathbf{H} = (H_1, H_2, H_3)^*$ is the magnetic

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field intensity vector, $S^{\|}$ and $S^{\perp}$ are the longitudinal and transverse entropies respectively. Moreover, the following thermodynamic identity [11] holds:

$$dE = T^{\|}dS^{\|} + T^{\perp}dS^{\perp} + \frac{p^{\|}}{\rho^2}d\rho - \frac{p^{\|}-p^{\perp}}{\rho w}dw, \tag{1.2}$$

where $E$ is the internal energy, $T^{\|}$ and $T^{\perp}$ are the longitudinal and transverse temperatures. It follows from (1.2) that

$$T^{\|} = E_{S^{\|}}, \quad T^{\perp} = E_{S^{\perp}}, \quad p^{\|} = \rho^2 E_\rho, \quad \frac{p^{\|}-p^{\perp}}{\rho w} = -E_w.$$

Thus, adding the plasma state equation to system (1.1)

$$E = E(\rho, S^{\|}, S^{\perp}, w),$$

we will close it. System (1.1) may be considered as an appropriate system for determining the components of the vector

$$\mathbf{U} = \begin{pmatrix} \rho \\ \mathbf{v} \\ \mathbf{H} \\ S^{\|} \\ S^{\perp} \end{pmatrix}.$$

The following condition, which is obligatory in magnetic hydrodynamics, should be included in (1.1):

$$\operatorname{div} \mathbf{H} = 0. \tag{1.3}$$

In essence, this is an additional requirement imposed on the initial data for system (1.1): $\mathbf{U}|_{t=0} = \mathbf{U}_0(\mathbf{x})$. This statement becomes obvios if we operate by div on $\mathbf{H}_t - \operatorname{rot}(\mathbf{v}\times\mathbf{H}) = 0$, assuming that condition (1.3) holds at $t = 0$.

Finally, we âlso add to (1.1) the energy conservation law

$$(\mathcal{E}_0)_t + \operatorname{div} \mathcal{E} = 0, \tag{1.4}$$

where

$$\mathcal{E}_0 = \rho E + \rho\frac{\omega^2}{2} + \frac{w^2}{8\pi}; \quad \omega = |\mathbf{v}|; \quad \mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)^* = \rho\mathbf{v}(E + \frac{\omega^2}{2}) + \frac{1}{4\pi}\mathbf{H}\times(\mathbf{v}\times\mathbf{H}) + p^{\perp}\mathbf{v} + (p^{\|} - p^{\perp})\mathbf{b}(\mathbf{b}, \mathbf{v}).$$

Note that this is the law (1.4) that was used in [12] in the symmetrization of Eqs. (1.1). The system can be rewritten, according to [12], in the symmetric form

$$A_0(\mathbf{U})\mathbf{U}_t + \sum_{k=1}^{3} A_k(\mathbf{U})\mathbf{U}_{x_k} = 0 \tag{1.5}$$

where $A_\alpha$ ($\alpha = \overline{0,3}$) are the symmetric matrices described in [12].

Further, let the plasma state equation be given by [11]

$$E = \frac{p^{\perp}}{\rho} + \frac{p^{\|}}{2\rho} \tag{1.6}$$

(the analog of the polytropic gas model for plasma). Then,

$$S^{\|} = \frac{c}{2}\ln\left(\frac{p^{\|}w^2}{\rho^3}\right), \quad S^{\perp} = c\ln\left(\frac{p^{\perp}}{\rho w}\right), \quad T^{\|} = \frac{p^{\|}}{c\rho}, \quad T^{\perp} = \frac{p^{\perp}}{c\rho}, \tag{1.7}$$

where $c$ is constant. Note also that last two equations in (1.1) are rewritten in this case as

$$\frac{d}{dt}\left(\frac{p^{\|}w^2}{\rho^3}\right) = 0, \qquad \frac{d}{dt}\left(\frac{p^{\perp}}{\rho w}\right) = 0.$$

Here $\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}, \nabla)$; $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)^*$.

As is known, surfaces of strong discontinuities (shock waves, rotational discontinuity, and so on) may appear in collisionless magnetized plasma. Let us consider the piecewise smooth solutions to system (1.1) in which the smooth pieces are separated by the strong discontinuity surface given by the equation

$$\tilde{f}(t, \mathbf{x}) = f(t, \mathbf{x}') - x_1 = 0, \qquad \mathbf{x}' = (x_2, x_3).$$

On the strong discontinuity surface, the following relations are valid [2]:

$$\begin{gathered}[j] = 0, \quad [H_N] = 0, \quad [jv_N + \tilde{\mathcal{P}}] = 0, \qquad j[v_{\tau_i}] = -H_N\left[\frac{H_{\tau_i}}{w^2}\mathcal{P}\right] \quad (i = 1, 2), \\ H_N[v_{\tau_i}] = j[VH_{\tau_i}] \quad (i = 1, 2), \qquad \left[V\mathcal{E}_0 j + \mathcal{P}^{\perp}v_N + \frac{\mathcal{P}}{w^2}H_N(\mathbf{H}, \mathbf{v})\right] = 0, \quad j[S^{\perp}] = 0,\end{gathered} \tag{1.8}$$

where the usual notation is used: $[F] = F - F_\infty$ [$F$ is a value ahead ofthe shock (as $\tilde{f} \to -0$) and $F_\infty$ is a value behind the shock (as $\tilde{f} \to +0$)]. Here, $j = \rho(v_N - D_N)$ is the mass flux across the shock, $v_N = (\mathbf{v}, \mathbf{N})$; $\mathbf{N} = \frac{1}{|\nabla \tilde{f}|}(-1, f_{x_2}, f_{x_3})^*$ is a normal to the strong discontinuity surface,

$$|\nabla \tilde{f}| = \sqrt{1 + f_{x_2}^2 + f_{x_3}^2}; \qquad D_N = \frac{-f_t}{|\nabla \tilde{f}|};$$

$H_N = (\mathbf{H}, \mathbf{N})$; $v_{\tau_1} = (\mathbf{v}, \boldsymbol{\tau}_1)$ and so on;

$$\boldsymbol{\tau}_1 = (f_{x_2}, 1, 0)^*; \quad \boldsymbol{\tau}_2 = (f_{x_3}, 0, 1)^*; \quad (\boldsymbol{\tau}_i, \mathbf{N}) = 0 \qquad (i = 1, 2);$$

$$\tilde{\mathcal{P}} = \mathcal{P}^{\perp} + \frac{p^{\|} - p^{\perp}}{w^2} H_N^2;$$

$V = 1/\rho$ is the specific weight. Note that we use the standard closure of the strong discontinuity relations for anisotropic plasma, namely, the last (closing) relation in (1.8) is the condition of the first adiabatic invariant conservation [13–15].

A detailed classification of strong discontinuities in anisotropic magnetic hydrodynamics is given in [2]. In the context of the present work, of interest are the shock waves for which the following conditions hold:

$$j \neq 0, \qquad [\rho] \neq 0. \tag{1.9}$$

In this case, the relation $[v_N]^2 + [V][\tilde{\mathcal{P}}] = 0$ can be used instead of the third condition in (1.8), and the equality (the analog of the Hugoniot adiabat in gas dynamics)

$$[E] + \langle p^{\|}\rangle[V] - \left\langle \frac{|\mathbf{H}_\sigma|^2}{w^2}(p^{\|} - p^{\perp})\right\rangle[V] + \left\langle \frac{p^{\|} - p^{\perp}}{w^2}|\mathbf{H}_\sigma|\right\rangle[V|\mathbf{H}_\sigma|] + \frac{1}{16\pi}[V][|\mathbf{H}_\sigma|]^2 = 0$$

[$\langle F\rangle = (F + F_\infty)/2$, and $\mathbf{H}_\sigma$ is the component of the vector $\mathbf{H}$ tangent to the shock surface] can be used instead of the next to last condition in (1.8).

**2. Linearization.** Linearizing system (1.5) with respect to a constant solution to system (1.1)

$$\mathbf{U} = \widehat{\mathbf{U}} = (\hat{\rho}, \hat{\mathbf{v}}^*, \widehat{\mathbf{H}}^*, \hat{S}^{\|}, \hat{S}^{\perp})^*,$$

we obtain the linear system

$$\tilde{A}_0(\delta\mathbf{U})_t + \sum_{k=1}^{3} \tilde{A}_k(\delta\mathbf{U})_{x_k} = 0. \tag{2.1}$$

Here $\widetilde{A}_\alpha = A_\alpha(\widehat{\mathbf{U}})$ $(\alpha = \overline{0,3})$; $\delta\mathbf{U} = (\delta\rho, \delta\mathbf{v}^*, \delta\mathbf{H}^*, \delta S^{\|}, \delta S^{\perp})^*$ is the vector of small disturbances. In the subsequent discussion we denote the vector $\delta\mathbf{U}$ by $\mathbf{U}$ again.

For the entropy disturbances, the equalities are derived from (1.7):

$$S^{\|} = \frac{1}{\widehat{T}^{\|}\widehat{\rho}}\Big\{\frac{1}{2}p^{\|} + \frac{\widehat{p}^{\|}}{\widehat{w}^2}(\widehat{\mathbf{H}}, \mathbf{H}) - \frac{3}{2}\frac{\widehat{p}^{\|}}{\widehat{\rho}}\rho\Big\}, \quad S^{\perp} = \frac{1}{\widehat{T}^{\perp}\widehat{\rho}}\Big\{p^{\perp} - \frac{\widehat{p}^{\perp}}{\widehat{w}^2}(\widehat{\mathbf{H}}, \mathbf{H}) - \frac{\widehat{p}^{\perp}}{\widehat{\rho}}\rho\Big\}.$$

Taking them in to account, it is possible to write system (2.1) in the dimensionless form

$$\begin{gathered} L\rho + \operatorname{div}\mathbf{v} = 0, \qquad L\mathbf{v} + \nabla\mathcal{P}^{\perp} - \bar{r}^2(\mathbf{h}, \nabla)\mathbf{H} + \frac{\mathbf{h}}{q^2}(\mathbf{h}, \nabla(\bar{p}p^{\|} - p^{\perp} + 2(\bar{r}^2 - 1)H_h)) = 0, \\ L\mathbf{H} - \operatorname{rot}(\mathbf{v}\times\mathbf{h}) = 0, \quad Lp^{\|} + \operatorname{div}\mathbf{v} + \frac{2}{q^2}(\mathbf{h}, \nabla v_h) = 0, \quad Lp^{\perp} + 2\operatorname{div}\mathbf{v} - \frac{1}{q^2}(\mathbf{h}, \nabla v_h) = 0, \end{gathered} \tag{2.2}$$

where the coordinate $x_k$ $(k = 1, 2, 3)$, time $t$, the components of the vectors $\mathbf{v}$ and $\mathbf{H}$, the values of $p^{\|}$, $p^{\perp}$ and $\rho$ are normalized to the characteristic parameters: $\widehat{l}$, $\widehat{l}/\widehat{c}$, $\widehat{c}$, $\widehat{c}\sqrt{4\pi\widehat{\rho}}$, $\widehat{p}^{\|}$, $\widehat{p}^{\perp}$, $\widehat{\rho}$; $\widehat{l}$ is the characteristic length; $\widehat{c} = (\widehat{p}^{\perp}/\widehat{\rho})^{1/2}$ is the sound velocity in plasma;

$$L = \tau + (\mathbf{M}, \nabla); \quad \tau = \frac{\partial}{\partial t}; \quad \nabla = (\xi_1, \xi_2, \xi_3)^*; \quad \xi_k = \frac{\partial}{\partial x_k} \quad (k = 1, 2, 3);$$

$$\mathbf{M} = \frac{\widehat{\mathbf{v}}}{\widehat{c}} = (\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3)^*; \quad \mathcal{P}^{\perp} = p^{\perp} + H_h; \quad H_h = (\mathbf{h}, \mathbf{H}); \quad v_h = (\mathbf{h}, \mathbf{v});$$

$$\bar{r} = \frac{(\bar{p}_2 - \bar{p})^{1/2}}{q}; \; q = |\mathbf{h}|; \; \mathbf{h} = (h_1, h_2, h_3)^* = \frac{\widehat{\mathbf{H}}}{\widehat{c}\sqrt{4\pi\widehat{\rho}}}; \; \bar{p}_2 = 1 + q^2; \; \bar{p} = \frac{\widehat{p}^{\|}}{\widehat{p}^{\perp}}.$$

In the statement of mixed problems for the systems (2.1) and (2.2), it is necessary to know the number of the boundary conditions at $x_1 = 0$ which should coincide with number there of the positive characteristic values of matrix $\widetilde{A}_0^{-1}\widetilde{A}_1$ [16]. After calculations, we find the matrix characteristic values (see also [2]):

$$\lambda_{1,2,3}(\widetilde{A}_0^{-1}\widetilde{A}_1) = \widehat{v}_1, \quad \lambda_{4,5}(\widetilde{A}_0^{-1}\widetilde{A}_1) = \widehat{v}_1 \pm \widehat{c}_A, \quad \lambda_{6,7}(\widetilde{A}_0^{-1}\widetilde{A}_1) = \widehat{v}_1 \pm \widehat{c}_{\mathrm{M}}^-, \quad \lambda_{8,9}(\widetilde{A}_0^{-1}\widetilde{A}_1) = \widehat{v}_1 \pm \widehat{c}_{\mathrm{M}}^+. \tag{2.3}$$

Here $\widehat{c}_A = \widehat{c}h_1\bar{r}$ is the Alfven velocity (for the sake of definiteness, we assume that $h_1 \geqslant 0$),

$$\widehat{c}_{\mathrm{M}}^{\pm} = \widehat{c}\Big\{\frac{\bar{p}_2 + (2\bar{p} - 1)l^2 + 1}{2} \pm \Big[\Big(\frac{\bar{p}_2 - l^2(1 + 4\bar{p}) + 1}{2}\Big)^2 + l^2(1 - l^2)\Big]^{1/2}\Big\}^{1/2}$$

are high and slow magnetic sound velocities in plasma [17]; $l = h_1/q$. For the matrix $\widetilde{A}_0$ to be positive definite, as shown in [12, 18], the inequalities

$$\bar{p}_1 < \bar{p} < \bar{p}_2 \tag{2.4}$$

(where $\bar{p}_1 = 1/\bar{p}_2$) should be valid. In this case, system (2.1) is symmetric and $t$-hyperbolic (according to Friedrichs). Moreover, the values of $\widehat{c}_A$ and $\widehat{c}_{\mathrm{M}}^{\pm}$ satisfy the inequalities [12, 18]

$$\begin{aligned} &0 < \widehat{c}_{\mathrm{M}}^- < \widehat{c}_A < \widehat{c}_{\mathrm{M}}^+, && \text{if} \quad \bar{p}_1 < \bar{p} < \bar{p}_* < \bar{p}_2, \\ &0 < \widehat{c}_{\mathrm{M}}^- = \widehat{c}_A < \widehat{c}_{\mathrm{M}}^+, && \text{if} \quad \bar{p} = \bar{p}_*, \; \bar{p}_1 < \bar{p}_* < \bar{p}_2, \\ &0 < \widehat{c}_A < \widehat{c}_{\mathrm{M}}^- < \widehat{c}_{\mathrm{M}}^+, && \text{if} \quad \bar{p}_1 < \bar{p}_* < \bar{p} < \bar{p}_2 \end{aligned} \tag{2.5}$$

$$\Big(\bar{p}_* = \frac{q^4 + 3q^2 + 3}{4(2 + q^2)}\Big).$$

Finally, we linearize system (1.1) with respect to the discontinuous solution. Let us consider the piecewise constant solution

$$\mathbf{U}(t, \mathbf{x}) = \begin{cases} \widehat{\mathbf{U}}_\infty = (\widehat{\rho}_\infty, \widehat{\mathbf{v}}_\infty^*, \widehat{\mathbf{H}}_\infty^*, \widehat{S}_\infty^{\|}, \widehat{S}_\infty^{\perp})^*, & x_1 < 0, \\ \widehat{\mathbf{U}} = (\widehat{\rho}, \widehat{\mathbf{v}}^*, \widehat{\mathbf{H}}^*, \widehat{S}^{\|}, \widehat{S}^{\perp})^*, & x_1 > 0 \end{cases} \tag{2.6}$$

to system (1.1), satisfying relations (1.8) at $x_1 = 0$ provided that the shock front is stationary and defined by

$$\widehat{\rho}\widehat{v}_1 = \widehat{\rho}_\infty \widehat{v}_{1\infty} = \widehat{j}, \quad \widehat{H}_1 = \widehat{H}_{1\infty}, \quad \widehat{j}^2[\widehat{V}] + \left[\widehat{p}^\perp + \frac{1}{8\pi}(\widehat{H}_2^2 + \widehat{H}_3^2)\right] + \left[\frac{\widehat{p}^\| - \widehat{p}^\perp}{\widehat{w}^2}\right]\widehat{H}_1^2 = 0,$$

$$\widehat{j}[\widehat{v}_i] = -\left[\widehat{\mathcal{P}}\frac{\widehat{H}_2}{\widehat{w}^2}\right]\widehat{H}_1 \quad (i = 2,3), \qquad \widehat{H}_1[\widehat{v}_i] = \widehat{j}[\widehat{V}\widehat{H}_2] \quad (i = 2,3), \tag{2.7}$$

$$\widehat{j}\left[\widehat{E} + \widehat{p}^\perp\widehat{V} + \frac{\widehat{j}^2\widehat{V}^2}{2} + \frac{\widehat{v}_2^2 + \widehat{v}_3^2}{2} + \frac{\widehat{H}_2^2 + \widehat{H}_3^2}{4\pi}\widehat{V} + \frac{\widehat{p}^\| - \widehat{p}^\perp}{\widehat{w}^2}\widehat{H}_1^2\widehat{V}\right] + \left[\frac{\widehat{\mathcal{P}}}{\widehat{w}^2}(\widehat{H}_2\widehat{v}_2 + \widehat{H}_3\widehat{v}_3)\right]\widehat{H}_1 = 0, \quad \widehat{j}[\widehat{S}^\perp] = 0.$$

Here $\widehat{V} = 1/\widehat{\rho}$; $\widehat{\mathcal{P}} = \widehat{p}^\| - \widehat{p}^\perp - \widehat{w}^2/(4\pi)$; $\widehat{E} = \widehat{p}^\perp + \widehat{p}^\|\widehat{V}/2$. Linearizing Eqs. (1.1) and relations (1.8) with respect to piecewise constant solution (2.6) and taking account of (2.7), we obtain the mathematical formulation of the strong discontinuities stability problem in magnetic hydrodynamics with anisotropic pressure.

**Main Problem.** In the domain $t > 0$, $\mathbf{x} \in R^3_+$, a solution is sought to the system

$$\widetilde{A}_0\mathbf{U}_t + \sum_{k=1}^{3}\widetilde{A}_k\mathbf{U}_{x_k} = 0, \tag{2.8}$$

while in the domain $t > 0$, $\mathbf{x} \in R^3_-$, a solution is sought to the system

$$\widetilde{A}_{0\infty}\mathbf{U}_t + \sum_{k=1}^{3}\widetilde{A}_{k\infty}\mathbf{U}_{x_k} = 0, \tag{2.9}$$

$$[\widehat{v}_1\rho + J] = 0, \quad [I] = 0, \quad 2[\widehat{v}_1 J] + [\widehat{v}_1^2\rho + \Pi^\perp + \Pi_1] = 0,$$

$$\left[\left(\widehat{\rho}\widehat{v}_1^2 + \frac{\widehat{\mathcal{P}}\widehat{H}_1^2}{\widehat{w}^2}\right)F_{x_k} + \widehat{v}_k(\widehat{v}_1\rho + J) + \widehat{j}v_k + \frac{\widehat{\mathcal{P}}\widehat{H}_1}{\widehat{w}^2}H_k + \frac{\widehat{H}_k}{\widehat{w}^2}\Pi_2\right] = 0,$$

$$[\widehat{V}\widehat{H}_k J + \widehat{v}_1 H_k - \widehat{v}_k I - \widehat{H}_1 v_k] = 0, \quad k = 2,3, \tag{2.10}$$

$$\left[\left(\widehat{E} + \frac{\widehat{\omega}^2}{2} + \widehat{p}^\perp\widehat{V} + \frac{\widehat{w}^2}{4\pi}\widehat{V}\right)J + \widehat{\mathcal{P}}^\perp F_t + \widehat{v}_1\left\{2\Pi^\perp + \frac{\widehat{p}^\|}{2} + \frac{\widehat{\omega}^2}{2}\rho + \widehat{\rho}(\widehat{\mathbf{v}},\mathbf{v})\right\}\right.$$

$$\left. + \frac{\widehat{\mathcal{P}}\widehat{H}_1}{\widehat{w}^2}\{(\widehat{\mathbf{H}},\mathbf{v}) + (\widehat{\mathbf{v}},\mathbf{H})\} + \frac{(\widehat{\mathbf{H}},\widehat{\mathbf{v}})}{\widehat{w}^2}\Pi_2\right] = 0, \quad [(\widehat{v}_1\rho + J)\widehat{S}^\perp + \widehat{j}S^\perp] = 0,$$

and at $t = 0$ they must satisfy the initial data

$$\mathbf{U}|_{t=0} = \mathbf{U}_0(\mathbf{x}), \quad \mathbf{x} \in R^3_\pm, \qquad F|_{t=0} = F_0(\mathbf{x}'), \quad \mathbf{x}' \in R^2. \tag{2.11}$$

Here $F = F(t,\mathbf{x}') = \delta f(t,\mathbf{x}')$ is small displacement of the shock front, $\widetilde{A}_{\alpha\infty} = A_\alpha(\widehat{\mathbf{U}}_\infty)$ $(\alpha = \overline{0,3})$;

$$J = \widehat{\rho}(\widehat{v}_1 - \widehat{v}_2 F_{x_2} - \widehat{v}_3 F_{x_3} - F_t); \quad I = H_1 - \widehat{H}_2 F_{x_2} - \widehat{H}_3 F_{x_3};$$

$$\Pi^\perp = p^\perp + \frac{(\widehat{\mathbf{H}},\mathbf{H})}{4\pi}; \quad \Pi_1 = \frac{\widehat{H}_1}{\widehat{w}^2}\left\{\widehat{H}_1(p^\| - p^\perp) + 2(\widehat{p}^\| - \widehat{p}^\perp)\left(I - \frac{\widehat{H}_1(\widehat{\mathbf{H}},\mathbf{H})}{\widehat{w}^2}\right)\right\};$$

$$\Pi_2 = \widehat{H}_1(p^\| - p^\perp) + \widehat{\mathcal{P}}I - 2(\widehat{\mathbf{H}},\mathbf{H})\widehat{H}_1\frac{\widehat{p}^\| - \widehat{p}^\perp}{\widehat{w}^2}; \qquad \widehat{\mathcal{P}}^\perp = \widehat{p}^\perp + \frac{\widehat{w}^2}{8\pi}.$$

Note that, solving mixed problem (2.8)–(2.11), we also find the function $F = F(t,\mathbf{x}')$. To do this, one of the conditions (2.10) should be considered as an equation for determining the function $F$. Furthermore, if all characteristic values of the matrix $\widetilde{A}_{1\infty}$ (or $\widetilde{A}_{0\infty}^{-1}\widetilde{A}_{1\infty}$) are nonnegative, we assume, without losing generality, that $\mathbf{U}(t,\mathbf{x}) \equiv 0$ at $x_1 < 0$.

**3. Statement of the Problem on Fast Shock Wave Stability.** In anisotropic magnetic hydrodynamics similar to the usual magnetic one, fast and slow shock waves exist. We consider the case of fast shock waves.

Let the stationary discontinuity be a shock wave ($\widehat{j} \neq 0$, $[\widehat{\rho}] \neq 0$). Without loss of generality, we assume that $\widehat{v}_1$, $\widehat{H}_1 > 0$. The conditions

$$\widehat{v}_{1\infty} > \widehat{c}^{+}_{\mathrm{M}\infty}, \qquad \widehat{c}^{+}_{\mathrm{M}} > \widehat{v}_1 > \max\{\widehat{c}_A, \widehat{c}^{-}_{\mathrm{M}}\} \tag{3.1}$$

or

$$\widehat{v}_{1\infty} > \widehat{c}^{+}_{\mathrm{M}\infty}, \qquad \widehat{c}^{+}_{\mathrm{M}} > \max\{\widehat{c}_A, \widehat{c}^{-}_{\mathrm{M}}\} > \widehat{v}_1 > \min\{\widehat{c}_A, \widehat{c}^{-}_{\mathrm{M}}\} \tag{3.1$'$}$$

correspond to a fast shock wave. By virtue of the first condition from (1.1), taking into account (2.3) and (2.5), it follows that all characteristic values of the matrix $\widetilde{A}^{-1}_{0\infty}\widetilde{A}_{1\infty}$ are positive and, consequently, system (2.9) does not require a boundary condition at $x_1 = 0$, i.e., $\mathbf{U} \equiv 0$ at $x_1 < 0$. By the second condition in (3.1), the matrix $\widetilde{A}_0^{-1}\widetilde{A}_1$ has eight positive characteristic values, i.e., eight boundary conditions should be set for system (2.8). As a result, nine boundary conditions are required for fast shock wave evolution [18]. One of them is the equation for seeking $F$. Using the same line of reasoning, in the case where conditions (3.1$'$) are valid, we conclude that eight boundary conditions are necessary for wave evolution at $x_1 = 0$, with no boundary condition being required again for system (2.9). In this case, the last (closing) relation in (1.8) is redundant.

Let conditions (3.1) hold on stationary discontinuity. State mathematically the problem on fast shock wave stability in the plane case.

**Problem $\mathcal{F}$.** In the domain $t > 0$, $\mathbf{x}\in R^2_+$, the solution is sought to the system of equations [see also (2.2)]

$$\begin{gathered} L\rho + \operatorname{div}\mathbf{v} = 0, \quad L\mathbf{v} + \mathbf{b}(\mathbf{b}, \nabla(\bar{p}p^{\|} - p^{\perp})) + \frac{\bar{p}-1}{q}(\mathbf{b},\nabla)(\boldsymbol{\sigma}H_\sigma - \mathbf{b}H_b) + \nabla p^{\perp} - q(\xi_1 H_2 - \xi_2 H_1)\boldsymbol{\sigma} = 0, \\ LH_1 + q\xi_2 v_\sigma = 0, \quad LH_2 - q\xi_1 v_\sigma = 0, \quad Lp^{\|} + \operatorname{div}\mathbf{v} + 2(\mathbf{b},\nabla v_b) = 0, \quad Lp^{\perp} + 2\operatorname{div}\mathbf{v} - (\mathbf{b},\nabla v_b) = 0. \end{gathered} \tag{3.2}$$

At $t > 0$, $x_1 = 0$, $x_2\in R^1$, the solution should satisfy the boundary conditions

$$\begin{gathered} F_t + \mathrm{M}_2 F_{x_2} = \frac{1}{1-\bar{\rho}}\{v_1 + \mathrm{M}_1\rho - \bar{\rho}[\mathrm{M}_2]F_{x_2}\}, \quad H_1 = [h_2]F_{x_2}, \\ p^{\perp} + qH_b + l^2(\bar{p}p^{\|} - p^{\perp}) - 2(\bar{p}-1)\frac{lm}{q}H_\sigma - 2lq[h_2]F_{x_2} + 2\mathrm{M}_1 v_1 + \mathrm{M}_1^2\rho = 0, \\ \Big\{\bar{p}^{\perp} - 1 - (\bar{p}-1)m^2 + \frac{1}{2}[h_2^2] + \frac{\bar{p}_\infty - 1}{l^2+\chi^2 m^2}\bar{p}^{\perp}\chi^2 m^2 + \frac{[\mathrm{M}_2]^2}{1-\bar{\rho}}\bar{\rho}\Big\}F_{x_2} - \frac{[\mathrm{M}_2]}{1-\bar{\rho}}\bar{\rho}(v_1 + \mathrm{M}_1\rho)\mathrm{M}_1 v_2 \\ + (\bar{p} - \bar{p}_2)\frac{lH_2 + mH_1}{q} + lm\Big\{\bar{p}p^{\|} - p^{\perp} - 2\frac{\bar{p}-1}{q}H_b\Big\} = 0, \quad \mathrm{M}_1 H_2 = [h_2](F_t + \mathrm{M}_2 F_{x_2}) + qv_\sigma, \\ \Big\{-\Big(2\bar{p}^{\perp} + \frac{\bar{p}_\infty\bar{p}^{\perp}}{2} + (l^2 + m^2\chi^2)q^2 - \Big(2 + \frac{\bar{p}}{2} + q^2 + \frac{[|\mathrm{M}|^2]}{2}\Big)\bar{\rho}\Big)\frac{[\mathrm{M}_2]}{1-\bar{\rho}}\bar{\rho} - \Big(1 - \bar{p}^{\perp} + \frac{[h_2^2]}{2}\Big)\frac{\mathrm{M}_2 - \bar{\rho}\mathrm{M}_{2\infty}}{1-\bar{\rho}} \\ + \frac{l\mathrm{M}_{1\infty} + \chi m \mathrm{M}_{2\infty}}{l^2 + \chi^2 m^2}\chi m\Big((\bar{p}_\infty - 1)\bar{p}^{\perp} - (l^2 + \chi^2 m^2)q^2\Big) - (\bar{p} - \bar{p}_2)(\mathbf{b},\mathbf{M})m\Big\}F_{x_2} \\ + \Big\{1 + \bar{p}^{\perp} + \frac{\bar{p}_\infty\bar{p}^{\perp}}{2} + (l^2 + \chi^2 m^2)q^2 + \frac{[h_2^2]}{2} - \Big(2 + \frac{\bar{p}}{2} + q^2 + \frac{[|\mathrm{M}|^2]}{2}\Big)\rho\Big\}\frac{v_1 + \mathrm{M}_1\rho}{1-\bar{\rho}} \\ + \mathrm{M}_1\Big\{2p^{\perp} + 2qH_b + \bar{p}\frac{p^{\|}}{2} + (\mathbf{M},\mathbf{v}) - \Big(2 + \frac{\bar{p}}{2} + q^2\Big)\rho\Big\} + (\bar{p} - \bar{p}_2)\Big\{lv_b + \frac{l}{q}(\mathbf{M},\mathbf{H}) + (\mathbf{b},\mathbf{M})\frac{H_1}{q}\Big\} \\ + l(\mathbf{b},\mathbf{M})\Big\{\bar{p}p^{\|} - p^{\perp} - \frac{2}{q}H_b(\bar{p}-1)\Big\} = 0, \quad S^{\perp} = 0 \end{gathered} \tag{3.3}$$

and the initial data at $t = 0$

$$\mathbf{V}|_{t=0} = \mathbf{V}_0(\mathbf{x}), \quad \mathbf{x}\in R^2_+, \qquad F|_{t=0} = F_0(x_2), \quad x_2\in R^1. \tag{3.4}$$

Here

$$\mathbf{U}=(\rho,\mathbf{v}^*,\mathbf{H}^*,p^{\|},p^{\perp})^*;\quad \mathbf{b}=(l,m)^*;\quad m=\frac{h_2}{q};\quad \boldsymbol{\sigma}=(-m,l)^*;\quad \nabla=(\xi_1,\xi_2)^*;\quad L=\tau+\mathrm{M}_1\xi_1+\mathrm{M}_2\xi_2;$$

$$\mathrm{M}_{2\infty}=\frac{\widehat{v}_{2\infty}}{\widehat{c}};\quad h_{2\infty}=\frac{\widehat{H}_{2\infty}}{\widehat{c}\sqrt{4\pi\widehat{\rho}}};\quad \bar{\rho}=\frac{\widehat{\rho}_\infty}{\widehat{\rho}};\quad \chi=\frac{h_{2\infty}}{h_2};\quad \bar{p}^{\perp}=\frac{\widehat{p}^{\perp}_\infty}{\widehat{p}^{\perp}};\quad \bar{p}_\infty=\frac{\widehat{p}^{\|}_\infty}{\widehat{p}^{\perp}_\infty};\quad \mathbf{M}=(\mathrm{M}_1,\mathrm{M}_2)^*;$$

$$\mathrm{M}_{1\infty}=\frac{\widehat{v}_{1\infty}}{\widehat{c}}=\bar{v}_1\mathrm{M}_1;\quad \bar{v}_1=\frac{\widehat{v}_{1\infty}}{\widehat{v}_1};\quad H_b=(\mathbf{b},\mathbf{H});\quad H_\sigma=(\boldsymbol{\sigma},\mathbf{H});\quad v_b=(\mathbf{b},\mathbf{v});\quad v_\sigma=(\boldsymbol{\sigma},\mathbf{v});$$

$S^{\perp}$ is a small disturbance of the transverse entropy divided by $\widehat{p}^{\perp}/(\widehat{T}^{\perp}\widehat{\rho})$. Boundary conditions (3.3) are taken from general conditions (2.10) and written in dimensionless form. Moreover, if we take the Galilei transforms

$$t'=t,\quad x_1'=x_1,\quad x_2'=x_2-\mathrm{M}_2t,$$

then the operator $L$ in system (3.2) and the aggregate $(F_t+\mathrm{M}_2F_{x_2})$ in boundary conditions (3.3) become

$$L=\tau+\mathrm{M}_1\xi_1\quad\text{and}\quad F_t$$

(all primes are dropped).

For the transverse entropy disturbance, the equation in dimensionless form holds

$$S^{\perp}=p^{\perp}-\rho-H_b/q.$$

On the other hand, the function $S^{\perp}(t,\mathbf{x})$ is the solution to the mixed problem

$$\begin{aligned} &LS^{\perp}=0, && t>0, && \mathbf{x}\in R^2_+,\\ &S^{\perp}=0, && x_1=0, && x_2\in R^1,\\ &S^{\perp}|_{t=0}=S^{\perp}_0(\mathbf{x}), && \mathbf{x}\in R^2_+. && \end{aligned}\tag{3.5}$$

Assume, without loss of generality, that $S^{\perp}_0(\mathbf{x})\equiv 0,\ \mathbf{x}\in R^2_+$. Then, we can assume, taking into account (3.5), that $S^{\perp}\equiv 0,\ t>0,\ \mathbf{x}\in R^2_+$. In this case

$$\rho=p^{\perp}-H_b/q.\tag{3.6}$$

Thus, taking account of (3.6), it is possible to simplify the statement of the problem $\mathcal{F}$ in the following way: the first equation in system (3.2) and the last equation in boundary conditions (3.3) are eliminated, and equality (3.6) replaces $\rho$.

It is shown in [18] that

$$\operatorname{div}\mathbf{H}\equiv 0,\quad t>0,\quad \mathbf{x}\in R^2_+.\tag{3.7}$$

If conditions (3.1′) hold, the last equality in boundary conditions (3.3) should be eliminated in the problem $\mathcal{F}$, with equality (3.6) not being true.

**4. Study of Stationary Discontinuity.** We consider relations (3.7) for a fast shock wave ($\widehat{j}\neq 0$, $[\widehat{\rho}]\neq 0$) in plane case. We write them as

$$\begin{gathered}\bar{\rho}\bar{v}_1=1,\quad h_1=h_{1\infty},\quad \mathrm{M}_1^2\frac{\bar{\rho}-1}{\bar{\rho}}+1-\bar{p}^{\perp}+\frac{q^2m^2(1-\chi^2)}{2}+l^2\Big(\bar{p}-1-\frac{\bar{p}^{\perp}(\bar{p}_\infty-1)}{l^2+\chi^2m^2}\Big)=0,\\ \mathrm{M}_1[\mathrm{M}_2]-q^2lm(1-\chi)+lm\Big(\bar{p}-1-\frac{\bar{p}^{\perp}\chi(\bar{p}_\infty-1)}{l^2+\chi^2m^2}\Big)=0,\quad l[\mathrm{M}_2]=\mathrm{M}_1m\Big(1-\frac{\chi}{\bar{\rho}}\Big),\\ \mathrm{M}_1\Big\{2\Big(1-\frac{\bar{p}^{\perp}}{\bar{\rho}}\Big)+\frac{1}{2}\Big(\bar{p}-\frac{\bar{p}_\infty}{\bar{\rho}}\bar{p}^{\perp}\Big)+\frac{\mathrm{M}_1^2}{2}\Big(1-\frac{1}{\bar{\rho}^2}\Big)+\frac{[\mathrm{M}_2^2]}{2}+q^2m^2\Big(1-\frac{\chi^2}{\bar{\rho}}\Big)+l^2\Big(\bar{p}-1-\frac{\bar{p}^{\perp}(\bar{p}_\infty-1)}{\bar{\rho}(l^2+\chi^2m^2)}\Big)\Big\}\\ +lm\Big\{\mathrm{M}_2(\bar{p}-1)-\frac{(\bar{p}_\infty-1)\bar{p}^{\perp}\chi\mathrm{M}_{2\infty}}{l^2+\chi^2m^2}-q^2(\mathrm{M}_2-\chi\mathrm{M}_{2\infty})\Big\}=0,\qquad \bar{p}^{\perp}=\bar{p}\sqrt{l^2+\chi^2m^2},\end{gathered}\tag{4.1}$$

where $h_{1\infty}=\widehat{H}_{1\infty}/(\widehat{c}\sqrt{4\pi\widehat{\rho}})$; $\mathrm{M}_{2\infty}=\widehat{v}_{2\infty}/\widehat{c}$.

The evolution of a fast shock wave in two particular cases [parallel wave ($l = 1,\ m = 0$) and transverse wave ($l = 0,\ m = 1$)] is shown in [10]. Here we consider the general case ($0 < l < 1$), suggesting the pressure in the plasma to be high [$\widehat{p}^{\|} \gg \widehat{w}^2/(4\pi)$, $\widehat{p}^{\perp} \gg \widehat{w}^2/(4\pi)$], i.e, $q \ll 1$ [$q^2 = \widehat{w}^2/(4\pi\widehat{p}^{\perp})$, see also (2.4)].

As in the usual magnetic hydrodynamics, e.g., [19, 20], without loss of generality, the following conditions are assumed to be valid:

$$\widehat{S}^{\|} > \widehat{S}^{\|}_{\infty}, \quad \widehat{p}^{\|} > \widehat{p}^{\|}_{\infty} > 0, \quad \widehat{p}^{\perp} > \widehat{p}^{\perp}_{\infty} > 0, \quad \widehat{\rho} > \widehat{\rho}_{\infty} > 0, \quad \widehat{v}_{1\infty} > \widehat{v}_1 > 0, \quad \widehat{H}_2 > \widehat{H}_{2\infty}.$$

They can be rewritten as

$$\widehat{S}^{\|} > \widehat{S}^{\|}_{\infty}, \quad \bar{p}_{\infty}\bar{p}^{\perp} < \bar{p}, \quad 0 < \bar{p}^{\perp} < 1, \quad 0 < \bar{\rho} < 1, \quad \bar{v}_1 > 1, \quad 0 < \chi < 1. \tag{4.2}$$

We expand all values involved in (4.1) into a series in the small parameter $q$

$$\begin{gathered}\bar{\rho} = R_0 + R_1 q + R_2 q^2 + \ldots, \quad \bar{p}^{\perp} = P_0 + P_1 q + P_2 q^2 + \ldots, \quad \bar{\chi} = Z_0 + Z_1 q + Z_2 q^2 + \ldots, \\ \bar{p} = y_0 + y_1 q + y_2 q^2 + \ldots, \quad \bar{p}_{\infty} = w_0 + w_1 q + w_2 q^2 + \ldots, \quad [\mathrm{M}_2] = m_0 + m_1 q + m_2 q^2 + \ldots\end{gathered} \tag{4.3}$$

and so on. Let us consider the parameter

$$\mathrm{M}_0 = \widehat{v}_1/\widehat{c}^{+}_{\mathrm{M}},$$

which, in view of the second condition in (3.1), satisfies the inequalities

$$0 < \mathrm{M}_0 < 1.$$

Substituting expansions (4.3) into (4.1) and taking (2.4) into account, we obtain

$$R_0 = P_0 = Z_0 = y_0 = w_0 = 1, \quad m_0 = m_1, \quad R_1 = P_1 = Z_1 = y_1 = w_1 = 0.$$

Then, omitting detailed calculations, the first inequality in (3.1) can be rewritten as

$$\mathrm{M}_0^2 > \rho_0.$$

Here $\rho_0 = 1 + O(q^2)$; $\rho_0 < 1$. Consequently,

$$\mathrm{M}_0^2 = 1 + O(q^2). \tag{4.4}$$

In view of the equation for the high magnetic sound velocity, we obtain

$$\mathrm{M}_1^2 = \mathrm{M}_0^2 k, \qquad k = \omega + O(q^2),$$

where $\omega = 1 + \dfrac{l^2}{2} + \sqrt{1 - 4l^2 + \dfrac{21}{4}l^4}$. Then, taking into account (4.4), we have

$$\mathrm{M}_1^2 = \omega + O(q^2). \tag{4.5}$$

We find, using (4.5), from (4.1) that

$$R_k = \Big(1 - \frac{m^2}{\omega}\Big)P_i, \quad Z_i = \frac{P_i}{\omega}, \quad m_i = \frac{m}{l}\,\frac{\omega - 2 + l^2}{\sqrt{\omega}}P_i, \quad y_i - w_i = \frac{2 - l^2 - \omega}{l^2}P_i \qquad (i = 2,\ 3), \tag{4.6}$$

with $P_2 \leqslant 0$ in view of (4.2). If $P_2 = 0$, then (4.6) are valid at $i = 3, 4$ ($P_3 \leqslant 0$) and so on. Therefore, without loss of generality, we assume that $P_2 < 0$. Note that the inequality $\bar{v}_1 > 1$ holds by virtue of the first relation in (4.1). It is easy to check that the first and second inequalities in (4.2) are also valid.

It remains to verify the validity of the inequalities

$$\bar{p}_{1\infty} < \bar{p}_{\infty} < \bar{p}_{2\infty}. \tag{4.7}$$

Here $\bar{p}_{2\infty} = 1 + q^2\bar{p}^{\perp}/\bar{\rho}^2$; $\bar{p}_{1\infty} = 1/\bar{p}_{2\infty}$. It is easy to check that (4.7) holds if

$$|w_2| < 1, \quad |y_2| < 1. \tag{4.8}$$

This means that the parameters $y_2$, $w_2$, and $P_2$ should be chosen so that (4.8) is true.

Finally, note that at $q \ll 1$ the case (3.1′) is not true for a fast shock wave, i.e., the shock wave is not evolutionary.

**5. A Priori Estimate of the Solution to Problem $\mathcal{F}$.** Let us prove the well-posedness of problem $\mathcal{F}$ at $q \ll 1$ and $0 < l < 1$. In other words, prove the stability of a fast magnetohydrodynamic shock wave in anisotropic plasma at high pressure, taking into account the result obtained in [10] at $l = 0, 1$.

It is now convenient to rewrite boundary conditions (3.3) in view of (4.6) as

$$\begin{gathered} v_1 + dp^{\perp} = N_1\xi_2 F, \quad \tau F = \mu p^{\perp} + N_2\xi_2 F, \quad v_2 = \lambda_0\xi_2 F + \eta p^{\perp}, \\ p^{\|} = v p^{\perp} + N_3\xi_2 F, \quad H_2 = mq(1-\chi)\tau F + q v_{\sigma}, \quad H_1 = mq(1-\chi)\xi_2 F, \end{gathered} \tag{5.1}$$

where

$$d = 1 - \frac{m^2}{\omega} + O(q^2); \quad \mu = \frac{1}{P_2}\frac{3l^2 - \omega}{20l^2m^2 - \omega(5l^2+3)} + O(q^2);$$

$$\lambda_0 = -P_2\Big(\frac{1}{\omega} + \frac{(\omega - 2m^2)(\omega + (\omega - m^2)(7l^2+1))}{l^2(\omega - m^2)^2\omega}\Big)q^2 + O(q^3);$$

$$\eta = -\frac{m}{l}\frac{\omega - m^2 - 1}{\omega} + O(q^2) \qquad (\eta = O(q^2) \quad \text{at} \quad l = 1/\sqrt{3});$$

$$v = \frac{\omega - 2m^2}{l^2} + O(q^2); \quad N_i = O(q^2) \qquad (i = 1, 2, 3)$$

(here, we omit too cumbersome calculations). In system (5.1), we use normalization of the parameters other than normalization used in the statement of problem $\mathcal{F}$. The difference is that in the set of characteristic parameters, $\widehat{l}/\widehat{v}_1$ is taken instead of $\widehat{l}/\widehat{c}$ and $\widehat{v}_1$ instead of $\widehat{c}$.

The third and fourth equations in system (3.2) can be rewritten as

$$LH_b + qL_{\sigma}v_{\sigma} = 0, \qquad LH_{\sigma} - qL_b v_{\sigma} = 0, \tag{5.2}$$

where $L_b = (\mathbf{b}, \nabla)$; $L_{\sigma} = (\boldsymbol{\sigma}, \nabla)$. Then, with (3.7) taken into account, it follows from (5.2) that there exists the function $\Phi = \Phi(t, \mathbf{x})$ so that $H_b = -qL_{\sigma}\Phi$, $H_{\sigma} = qL_b\Phi$, $L\Phi = v_{\sigma}$.

We take into consideration the function $\Psi = \Psi(t, \mathbf{x})$ also:

$$\Psi = \frac{-\bar{p}p^{\|} + (1-\bar{p})L_{\sigma}\Phi + \frac{1}{2}p^{\perp}}{3\bar{p} - \frac{1}{2}}.$$

As follows from the last two equations in (3.2), the function $\Psi$ satisfies the equation $L\Psi = L_b v_b$. Thus, problem $\mathcal{F}$ can be restated.

**Problem $\mathcal{F}'$.** In the domain $t > 0$, $\mathbf{x} \in R^2_+$, we seek a solution to the system of equations

$$\begin{gathered} Lp^{\perp} + L_b v_b + 2L_{\sigma}v_{\sigma} = 0, \qquad \mathrm{M}_1^2 L v_b + \frac{1}{2}L_b p^{\perp} - \Big(3\bar{p} - \frac{1}{2}\Big)L_b\Psi = 0, \\ \mathrm{M}_1^2 L v_{\sigma} + L_{\sigma}p^{\perp} - (\bar{p}_2 - \bar{p})L_b^2\Phi - q^2L_{\sigma}^2\Phi = 0, \quad L\Phi = v_{\sigma}, \qquad L\Psi = L_b v_b, \end{gathered} \tag{5.3}$$

which at $t > 0$, $x_1 = 0$, $x_2 \in R^1$ satisfies the boundary conditions

$$v_1 + dp^{\perp} = N_1\xi_2 F, \;\; \tau F = \mu p^{\perp} + N_2\xi_2 F, \;\; v_2 = \lambda_0\xi_2 F + \eta p^{\perp}, \;\; \Psi = æ p^{\perp} + N_4\xi_2 F, \;\; \Phi = -m(1-\chi)F, \tag{5.4}$$

and at $t = 0$ satisfies the initial data. Here

$$æ = -\frac{2(\omega - m^2 - 1)}{5l^2} - \frac{1}{5} + O(q^2); \qquad N_4 = O(q^2).$$

System (5.3) can be rewritten in the symmetric $t$-hyperbolic (according to Friedrichs) form

$$A\mathbf{V}_t + B\mathbf{V}_{x_1} + C\mathbf{V}_{x_2} + \Omega\mathbf{V} = 0. \tag{5.5}$$

Here $\mathbf{V} = (p^{\perp}, v_b, v_\sigma, Q, R, \Psi, \Phi)^*$; $Q = L_b\Phi$; $R = L_\sigma\Phi$; $A = \mathrm{diag}\,(1/2,\ \mathrm{M}_1^2,\ \mathrm{M}_1^2,\ \bar{p}_2 - \bar{p},\ q^2, 3\bar{p} - 1/2,\ 1)$ is the diagonal matrix;

$$B = A + B_0, \qquad B_0 = \begin{pmatrix} 0 & \frac{1}{2}l & -m & 0 & 0 & 0 & 0 \\ \frac{1}{2}l & 0 & 0 & 0 & 0 & \left(\frac{1}{2} - 3\bar{p}\right) l & 0 \\ -m & 0 & 0 & (\bar{p}_2 - \bar{p})l & mq^2 & 0 & 0 \\ 0 & 0 & (\bar{p}_2 - \bar{p})l & 0 & 0 & 0 & 0 \\ 0 & 0 & mq^2 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{1}{2} - 3\bar{p}\right) l & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$C = \begin{pmatrix} 0 & \frac{1}{2}m & l & 0 & 0 & 0 & 0 \\ \frac{1}{2}m & 0 & 0 & 0 & 0 & \left(\frac{1}{2} - 3\bar{p}\right) m & 0 \\ l & 0 & 0 & (\bar{p}_2 - \bar{p})m & -lq^2 & 0 & 0 \\ 0 & 0 & (\bar{p}_2 - \bar{p})m & 0 & 0 & 0 & 0 \\ 0 & 0 & -lq^2 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{1}{2} - 3\bar{p}\right) m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$\Omega = (\omega_{ij})$ $(i, j = \overline{1, 7})$ is a matrix in which the element $\omega_{73} = -1$ and the other elements $\omega_{ij} = 0$. Note that, as follows from (2.4),

$$\bar{p}_2 - \bar{p} > 0, \quad 3\bar{p} - 1/2 > 0,$$

i.e, $A > 0$.

Now extend system (5.5) in the following way:

$$A_p(\mathbf{V}_p)_t + B_p(\mathbf{V}_p)_{x_1} + C_p(\mathbf{V}_p)_{x_2} + \Omega_p\mathbf{V}_p = 0, \tag{5.6}$$

where

$$\mathbf{V}_p = (\mathbf{V}^*, \tau\mathbf{V}^*, \xi_1\mathbf{V}^*, \xi_2\mathbf{V}^*, \tau^2\mathbf{V}^*, \tau\xi_1\mathbf{V}^*, \tau\xi_2\mathbf{V}^*, \xi_1^2\mathbf{V}^*, \xi_1\xi_2\mathbf{V}^*, \xi_2^2\mathbf{V}^*)^*;$$
$$A_p = \mathrm{diag}\,(A, A, A, A, A, A, A, A, A, A)$$

is a square-diagonal matrix and so on.

Writing the energy integral for system (5.6) in the differential form [3] and integrating it over the domain $R_+^2$, we obtain

$$\frac{d}{dt}J_0(t) - \int\limits_{R^1} (B_p\mathbf{V}_p, \mathbf{V}_p)\Big|_{x_1=0} dx_2 + \iint\limits_{R_+^2} ((\Omega_p + \Omega_p^*)\mathbf{V}_p, \mathbf{V}_p)\, d\mathbf{x} = 0. \tag{5.7}$$

Here

$$J_0(t) = \iint\limits_{R_+^2} (A_p\mathbf{V}_p, \mathbf{V}_p)\, d\mathbf{x}, \quad (A_p\mathbf{V}_p, \mathbf{V}_p) = (A\mathbf{V}, \mathbf{V}) + \ldots + (A\mathbf{V}_{x_2x_2}, \mathbf{V}_{x_2x_2});$$

$$(A\mathbf{V}, \mathbf{V}) = \frac{1}{2}(p^{\perp})^2 + \mathrm{M}_1^2 v_b^2 + \mathrm{M}_1^2 v_\sigma^2 + \frac{\bar{p}_2 - \bar{p}}{q^2} H_\sigma^2 + H_b^2 + \Phi^2 + \frac{\left(\bar{p}p^{\parallel} + \dfrac{1 - \bar{p}}{q} H_b - \dfrac{1}{2}p^{\perp}\right)^2}{3\bar{p} - \dfrac{1}{2}}$$

and so on. We assume that $(\mathbf{V}_p, \mathbf{V}_p)^{1/2} = |\mathbf{V}_p| \to 0$ as $x_1 \to \infty$ or $|x_2| \to \infty$.

The second and third terms in equality (5.7) are estimated at $x_1 = 0$ in view of (5.3) and (5.1). As a result, we write

$$\frac{d}{dt}J_0(t) - \int\limits_{R^1}\Big\{C_1((p^\perp)^2 + v_2^2 + (p_t^\perp)^2 + (p_{x_1}^\perp)^2 + (p_{x_2}^\perp)^2 + P)\Big|_{x_1=0} + C_q(\widetilde{F}^2 + \widetilde{F}_t^2 + \widetilde{F}_{x_2}^2)\Big\}dx_2 \leqslant C_2 J_0(t), \quad (5.8)$$

where $C_1$ and $C_2 > 0$ are positive constants, $P = (p_{tt}^\perp)^2 + (p_{tx_1}^\perp)^2 + (p_{tx_2}^\perp)^2 + (p_{x_1x_1}^\perp)^2 + (p_{x_1x_2}^\perp)^2 + (p_{x_2x_2}^\perp)^2$; $C_q = O(q^2)$; $\widetilde{F} = F_{x_2x_2}$. Considering again system (5.3) at $x_1 = 0$, after some cumbersome rearrangements, with the help of boundary conditions (5.4), we obtain the equality

$$\widetilde{F} = (a_1 p_t^\perp + a_2 p_{x_1}^\perp + a_3 p_{x_2}^\perp)\Big|_{x_1=0} \quad (a_i = O(1),\ i = \overline{1,3}).$$

Using this equation and invoking the property of the trace of a function from $W_2^1(R_+^2)$ along the line $x_1 = 0$ [21], we bring the inequality (5.8) to the following form:

$$\frac{d}{dt}J_0(t) - \widetilde{C}_1 \int\limits_{R^1} P|_{x_1=0}\, dx_2 \leqslant \widetilde{C}_2 J_0(t) \quad (5.9)$$

($\widetilde{C}_1, \widetilde{C}_2 > 0$ are constant).

Now we come to the second stage of constructing the extended system. After simple calculations, we find from system (5.3) that the functions $p^{\|}$, $\Phi$, and $\Psi$ satisfy the following equations:

$$\mathrm{M}^2 L^2 p^\perp - \xi_1^2 p^\perp - \gamma \xi_2^2 p^\perp + S = 0; \quad (5.10)$$

$$\mathrm{M}^2 L^2 \Phi + \frac{1}{\widetilde{k}}\left\{L_\sigma p^\perp - (\bar{p}_2 - \bar{p})L_b^2\Phi - q^2 L_\sigma^2 \Phi\right\} = 0. \quad (5.11)$$

Here $\mathrm{M}^2 = \mathrm{M}_1^2/\widetilde{k} = 1 - \delta q^2 + O(q^3)$; $\widetilde{k} = k\rho_1 = \omega + O(q^2)$; $\rho_1 = 1 + O(q^2)$; $\delta > 0$ is an arbitrary constant which is determined finally by choosing the parameter $\widetilde{k}$,

$$S = \frac{1}{\widetilde{k}}\left\{\left(3\bar{p} - \frac{1}{2}\right)L_b^2\Psi + \left(\widetilde{k} - 2 + \frac{3l^2}{2}\right)\xi_1^2 p^\perp + 3lm\xi_1\xi_2 p^\perp + 2L_\sigma((\bar{p}_2 - \bar{p})L_b^2\Phi + q^2 L_\sigma^2\Phi)\right\};$$

$$\gamma = \frac{1 + 3l^2}{2\widetilde{k}} = \frac{1 + 3l^2}{2\omega} + O(q^2).$$

We rewrite, following [3, 21] (see also [9]), Eq. (5.10):

$$(\widetilde{L}_1^2 - L_2^2 - \widetilde{L}_3^2)p^\perp + \beta^2 S = 0. \quad (5.12)$$

Here $\widetilde{L}_1 = \mathrm{M}L_1$; $L_1 = \tau$; $L_2 = \beta^2\xi_1 - \mathrm{M}^2 L_1$; $\widetilde{L}_3 = \beta L_3$; $L_3 = \sqrt{\gamma}\xi_2$; $\beta = \sqrt{1 - \mathrm{M}^2} = \sqrt{\delta}q + O(q^2)$. If the function $p^\perp(t, \mathbf{x})$ satisfies (5.12), then the vector

$$\mathbf{W} = (\mathbf{Y}_1^*, \mathbf{Y}_2^*, \mathbf{Y}_3^*)^* \quad (\mathbf{Y}_1 = \widetilde{L}_1\mathbf{Y}, \quad \mathbf{Y}_2 = L_2\mathbf{Y}, \quad \mathbf{Y}_3 = \widetilde{L}_3\mathbf{Y}, \quad \mathbf{Y} = \widetilde{\nabla}p^\perp, \quad \widetilde{\nabla} = (\widetilde{L}_1, L_2, \widetilde{L}_3)^*)$$

satisfies the following system [3]:

$$\{\hat{A}\widetilde{L}_1 - \hat{B}L_2 - \hat{C}\widetilde{L}_3\}\mathbf{W} + \beta^2 \begin{pmatrix} \mathcal{K} \\ \mathcal{L} \\ \mathcal{M} \end{pmatrix} \widetilde{\nabla}S = 0; \quad (5.13)$$

where

$$\widehat{A} = \begin{pmatrix} \mathcal{K} & \mathcal{L} & \mathcal{M} \\ \mathcal{L} & \mathcal{K} & i\mathcal{N} \\ \mathcal{M} & -i\mathcal{N} & \mathcal{K} \end{pmatrix}, \qquad \widehat{B} = \begin{pmatrix} \mathcal{L} & \mathcal{K} & i\mathcal{N} \\ \mathcal{K} & \mathcal{L} & \mathcal{M} \\ -i\mathcal{N} & \mathcal{M} & -\mathcal{L} \end{pmatrix}, \qquad \widehat{C} = \begin{pmatrix} \mathcal{M} & -i\mathcal{N} & \mathcal{K} \\ i\mathcal{N} & -\mathcal{M} & \mathcal{L} \\ \mathcal{K} & \mathcal{L} & \mathcal{M} \end{pmatrix},$$

here $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are arbitrary Hermitian matrices of order three. Returning to the differential operators $\tau$, $\xi_1$, $\xi_2$ in (5.13), we obtain the system

$$\left\{D\tau - \beta^2\widehat{B}\xi_1 - \beta\sqrt{\gamma}\widehat{C}\xi_2\right\}\mathbf{W} + \beta^2 \begin{pmatrix} \mathcal{K} \\ \mathcal{L} \\ \mathcal{M} \end{pmatrix} \widetilde{\nabla} S = 0 \qquad (D = \mathrm{M}(\widehat{A} + \mathrm{M}\widehat{B})). \tag{5.14}$$

The following relations [3] are valid:

$$\widehat{A} = T_0^*\{I_2 \times \widetilde{H}\}T_0, \quad \widehat{B} = T_0^*\left\{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \times \widetilde{H}\right\}T_0, \quad \widehat{C} = T_0^*\left\{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \times \widetilde{H}\right\}T_0. \tag{5.15}$$

Here

$$T_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \times I_3; \qquad \widetilde{H} = \begin{pmatrix} \mathcal{K} - \mathcal{M} & -\mathcal{L} - i\mathcal{N} \\ -\mathcal{L} + i\mathcal{N} & \mathcal{K} + \mathcal{M} \end{pmatrix};$$

$I_2 \times \widetilde{H}$ is the Kronecker product of the matrices $I_2$ and $\widetilde{H}$ and so on; $I_2$ is unit matrix of order 2 and so on. By (5.15)

$$D = \mathrm{M}T_0^*\left\{\begin{pmatrix} 1 & -\mathrm{M}_0 \\ -\mathrm{M}_0 & 1 \end{pmatrix} \times \widetilde{H}\right\}T_0. \tag{5.16}$$

Let us obtain the boundary conditions for system (5.13). For this purpose, multiply scalarly system (5.3) by the vector $(\mathrm{M}^2\tau, -l\tau/\tilde{k}, 2m\tau/\tilde{k}, 0, 0)^*$. Considering the obtained expression at $x_1 = 0$ and using boundary conditions (5.4), we obtain the relation

$$\left\{\mathrm{M}^2(1 + d\rho_2)\tau^2 - \beta^2\rho_3\tau\xi_1 + \mathrm{M}^2\lambda(\sqrt{\gamma}\xi_2)^2 + N_5\tau\xi_2\right\}p^\perp = 0, \quad x_1 = 0, \tag{5.17}$$

where $\lambda = \dfrac{1}{\gamma}\lambda_0\mu + O(q^2)$; $\rho_i = 1 + O(q^2)$ $(i = 2, 3)$; $N_5 = O(q^2)$. Consider also Eq. (5.10) at $x_1 = 0$. Using (5.4), we put it in the following form:

$$(\rho_4\widetilde{L}_1^2 - \rho_5 L_2^2 - \rho_6\widetilde{L}_3^2)p^\perp + (N_6\widetilde{L}_1 L_2 + N_7\widetilde{L}_1 L_3 + N_8 L_2\widetilde{L}_3)p^\perp = 0, \qquad x_1 = 0. \tag{5.18}$$

Here $\rho_i = 1 + O(q^2)$ $(i = \overline{4,6})$; $N_i = O(q^2)$ $(i = \overline{6,8})$. Taking into account (5.17) and (5.18), we take the expressions [3, 9] below as the boundary conditions for system (5.13) at $x_1 = 0$

$$\rho_4\widetilde{L}_1(\widetilde{L}_1 p^\perp) - \rho_5 L_2(L_2 p^\perp) - \rho_6\widetilde{L}_3(\widetilde{L}_3 p^\perp) + \alpha\{\rho_7\widetilde{L}_1(L_2 p^\perp) - \rho_8 L_2(\widetilde{L}_1 p^\perp)\} + N_7\widetilde{L}_1(\widetilde{L}_3 p^\perp) + N_8 L_2(\widetilde{L}_3 p^\perp) = 0,$$

$$\widetilde{L}_3(L_2 p^\perp) - L_2(\widetilde{L}_3 p^\perp) = 0,$$

$$\rho_9\widetilde{L}_1(L_2 p^\perp) - \rho_{10}\mathrm{M}dL_2(L_2 p^\perp) - \frac{\mathrm{M}}{\beta}\widetilde{m}\widetilde{L}_3(\widetilde{L}_3 p^\perp) + N_9\widetilde{L}_1(\widetilde{L}_3 p^\perp) + N_{10}L_2(\widetilde{L}_3 p^\perp) + N_{11}\widetilde{L}_1(\widetilde{L}_1 p^\perp) = 0,$$

and write them as

$$A_1\mathbf{Y}_1 + B_1\mathbf{Y}_2 + C_1\mathbf{Y}_3 = 0, \tag{5.19}$$

where

$$A_1 = \begin{pmatrix} \rho_4 & \alpha\rho_7 & N_7 \\ 0 & 0 & 0 \\ N_{11} & \rho_9 & N_9 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -\alpha\rho_8 & -\rho_5 & N_8 \\ 0 & 0 & -1 \\ 0 & -\rho_{10}\mathrm{M}d & N_{10} \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & -\rho_6 \\ 0 & 1 & 0 \\ 0 & 0 & -\dfrac{\mathrm{M}\widetilde{m}}{\beta} \end{pmatrix},$$

here $\alpha > 1$ is constant, $\widetilde{m} = \rho_{11}\beta d + \mathrm{M}^2\lambda/\beta$; $\rho_i = 1 + O(q^2)$ $(i = \overline{7,11})$; $N_i = O(q^2)$ $(i = \overline{9,11})$. It is easy to verify that $\lambda \hat{<} 0$ $(\mu < 0,\ \lambda_0 > 0)$. Choose a value of $\delta$ (i.e., $\widetilde{k}$) such that the coefficient

$$\widetilde{m} = \beta\Big(1 - \frac{m^2}{\omega} + \frac{\lambda^{(2)}}{\delta} + O(q^2)\Big)$$

becomes positive. Here $\lambda = \lambda^{(2)}q^2 + O(q^3)$; $\lambda^{(2)} < 0$; $1 - m^2/\omega > 0$.

Let

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_{\mathrm{I}} \\ \mathbf{\Lambda}_{\mathrm{II}} \end{pmatrix} = T_0\mathbf{W},$$

where

$$\mathbf{\Lambda}_{\mathrm{I}} = \begin{pmatrix} \mathbf{\Lambda}_1 \\ \mathbf{\Lambda}_2 \end{pmatrix}; \qquad \mathbf{\Lambda}_{\mathrm{II}} = \begin{pmatrix} \mathbf{\Lambda}_3 \\ \mathbf{\Lambda}_4 \end{pmatrix};$$

$\mathbf{\Lambda}_k$ $(k = \overline{1,4})$ are three-dimensional vectors. Since

$$\mathbf{Y}_1 = \frac{\sqrt{2}}{2}(\mathbf{\Lambda}_1 + \mathbf{\Lambda}_4), \quad \mathbf{Y}_2 = -\sqrt{2}\mathbf{\Lambda}_2 = -\sqrt{2}\mathbf{\Lambda}_3, \quad \mathbf{Y}_3 = \frac{\sqrt{2}}{2}(\mathbf{\Lambda}_4 - \mathbf{\Lambda}_1),$$

conditions (5.19) can be given as

$$\mathbf{\Lambda}_{\mathrm{I}} = G\mathbf{\Lambda}_{\mathrm{II}} \tag{5.20}$$

$\Big(G = \begin{pmatrix} G_1 & -G_2 \\ I_3 & 0 \end{pmatrix},\ G_1 = 2(A_1 - C_1)^{-1}B_1,\ G_2 = (A_1 - C_1)^{-1}(A_1 + C_1)\Big).$

Let all characteristic numbers of the matrix $G$ lie strictly in the left half-plane, i.e., $\operatorname{Re}\lambda_j(G) < 0$, $j = \overline{1,6}$. The latter is valid if $\widetilde{m} > 0,\ \lambda < 0$ [3]! We set up the Lyapunov equation

$$G^*\widetilde{H} + \widetilde{H}G = -G_0 \tag{5.21}$$

to find the matrix $\widetilde{H}$ from (5.15). As is known (see, e.g., [22]), Eq. (5.21) has the unique solution

$$\widetilde{H} = \begin{pmatrix} \widetilde{H}_1 & \widetilde{H}_2 \\ \widetilde{H}_2^* & \widetilde{H}_3 \end{pmatrix} > 0, \qquad \widetilde{H}_1 = \widetilde{H}_1^*, \qquad \widetilde{H}_3 = \widetilde{H}_3^*$$

at any real symmetric positive-definite matrix $G_0$. The matrix $\widetilde{H}$ is also real and symmetric, and matrices $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are found in the following way:

$$\mathcal{K} = \frac{1}{2}(\widetilde{H}_1 + \widetilde{H}_3), \qquad \mathcal{M} = \frac{1}{2}(\widetilde{H}_3 - \widetilde{H}_1), \qquad \mathcal{L} = -\frac{1}{2}(\widetilde{H}_2 + \widetilde{H}_2^*), \qquad i\mathcal{N} = \frac{1}{2}(\widetilde{H}_2^* - \widetilde{H}_2).$$

Since $\widetilde{H} > 0$, we have $D > 0$ [see (5.16)].

Write for system (5.14) the energy integral in the differential form [3]

$$(D\mathbf{W},\mathbf{W})_t - \beta^2(\widehat{B}\mathbf{W},\mathbf{W})_{x_1} - \beta\sqrt{\gamma}(\widehat{C}\mathbf{W},\mathbf{W})_{x_2} + \beta^2\{2(\mathbf{Y}_1,\mathcal{K}\widetilde{\nabla}S) + 2(\mathbf{Y}_2,\mathcal{L}\widetilde{\nabla}S) + 2(\mathbf{Y}_3,\mathcal{M}\widetilde{\nabla}S)\} = 0. \tag{5.22}$$

Taking account of system (5.3) and equality (5.12), we can write the term in braces in (5.22) as

$$\{\cdots\} = \tau\Omega_0 + \xi_1\Omega_1 + \xi_2\Omega_2. \tag{5.23}$$

Since the equations for $\Omega_\alpha$ $(\alpha = \overline{0,2})$ are cumbersome, they are omitted here.

Assuming $|\mathbf{W}| \to 0$ as $x_1 \to \infty$ or $|x_2| \to \infty$ and so on, we integrate identity (5.22), taking into account (5.3), over the domain $R_+^2$ to obtain

$$\frac{d}{dt} J_1(t) + \beta^2 \int_{R^1} \{(\hat{B}\mathbf{W}, \mathbf{W}) - \Omega_1\}\Big|_{x_1=0} dx_2 = 0. \tag{5.24}$$

Here

$$J_1(t) = \iint_{R_+^2} \{(D\mathbf{W}, \mathbf{W}) + \beta^2 \Omega_0\}\, d\mathbf{x}.$$

Note that the quadratic form

$$(\hat{B}\mathbf{W}, \mathbf{W})\Big|_{x_1=0} = (G_0 \Lambda_{\rm II}, \Lambda_{\rm II})\Big|_{x_1=0}$$

is positive definite, as follows from (5.15) and (5.20). Moreover, since

$$\Lambda_{\rm II} = \frac{\sqrt{2}}{2} \begin{pmatrix} -\mathbf{Y}_2 \\ \mathbf{Y}_1 + \mathbf{Y}_3 \end{pmatrix},$$

then

$$(\hat{B}\mathbf{W}, \mathbf{W})\Big|_{x_1=0} > C_3 \Big\{ (\tilde{L}_1^2 p^\perp)^2 + (\tilde{L}_1 L_2 p^\perp)^2 + (\tilde{L}_1 \tilde{L}_3 p^\perp)^2$$
$$+ (L_2^2 p^\perp)^2 + (L_2 \tilde{L}_3 p^\perp)^2 + (\tilde{L}_3^2 p^\perp)^2 \Big\}\Big|_{x_1=0} > \tilde{C}_3 \beta^8 P\Big|_{x_1=0}, \tag{5.25}$$

where $C_3$ and $\tilde{C}_3 > 0$ are positive constants independent of $q$ and determined finally by the norm of the matrix $G_0$. Note also that it is possible to obtain the inequality

$$-\beta^2 \Omega_1|_{x_1=0} > N_{12} P|_{x_1=0} \qquad (N_{12} = O(q^2)) \tag{5.26}$$

by using system (5.3) and boundary conditions (5.4) at $x_1 = 0$.

Because $q$ is infinitesimal, the quadratic form

$$(A_p \mathbf{V}_p, \mathbf{V}_p) + (D\mathbf{W}, \mathbf{W}) + \beta^2 \Omega_0$$

is positive definite ($A_p > 0,\ D > 0,\ \beta^2 = O(q^2)$). Therefore, adding (5.24) and (5.25) and taking into account (5.26) and the choice of the matrix $G_0$ [see (5.25)], we can obtain the positive-definite form

$$\mathcal{A} = \{\beta^2 (\hat{B}\mathbf{W}, \mathbf{W}) - \beta^2 \Omega_1 - \tilde{C}_1 P\}\Big|_{x_1=0} > (\beta^{10} \tilde{C}_3 - \tilde{C}_1 + N_{12}) P\Big|_{x_1=0} > 0.$$

For example, we can choose $G_0$ so that $\tilde{C}_3 = O(q^{-11})$. As a result, the following inequality is obtained:

$$\frac{d}{dt} J(t) \leqslant C_5 J(t), \qquad t > 0,$$

where $J(t) = J_0(t) + J_1(t)$; $C_5 > 0$ is a constant independent of $q$. From this inequality, the a priori estimate for problem $\mathcal{F}'$ follows:

$$J(t) \leqslant e^{C_5 t} J(0), \qquad t > 0. \tag{5.27}$$

This proves that the mixed problem $\mathcal{F}'$ is well-posed.

Let initial data (3.4) be such that

$$H_k|_{t=0} = q\varphi_k(\mathbf{x}), \qquad \mathbf{x} \in R_+^2, \qquad k = 1, 2.$$

Then, the function $\Phi_0(\mathbf{x})$ $(= \Phi|_{t=0})$ is found as a solution to the Dirichlet problem for the Poisson equation

$$\Delta\Phi_0 = \xi_1\varphi_2 - \xi_2\varphi_1, \quad \mathbf{x} \in R^2_+, \qquad \Phi_0\Big|_{x_1=0} = -m(1-\chi)F_0(x_2), \quad x_2 \in R^1.$$

Assume also that the functions $\varphi_k(\mathbf{x})$, $k = 1, 2$, $\mathbf{x} \in R^2_+$, are finite, with compact carriers lying in the bounded domain $\Omega \subset R^2_+$ with the smooth boundary $\partial\Omega$. Then we define the function $\Phi_0(\mathbf{x})$ as follows. In the domain $\bar{R}^2_+\backslash\Omega$ $\Phi_0(\mathbf{x}) \equiv -m(1-\chi)F_0(x_2)$ and in the domain $\Omega$, it is found as a solution to the Dirichlet problem

$$\Delta\Phi_0 = \xi_1\varphi_2 - \xi_2\varphi_1, \quad \mathbf{x} \in \Omega, \qquad \Phi_0|_{\partial\Omega} = -m(1-\chi)F_0(x_2).$$

Then, the following estimate is valid for the function $\Phi_0(\mathbf{x})$ [23]:

$$\|\Phi_0\|_{W^2_2(R^2_+)} \leqslant C_6\{\, \|\varphi_1\|_{W^2_2(R^2_+)} + \|\varphi_2\|_{W^2_2(R^2_+)} + \|F_0\|_{W^2_2(R^1)}\}.$$

Here $C_6 > 0$ is a positive constant independent of $\varphi_{1,2}$, $F_0$. From the last inequality by boundary conditions (5.4) we derive the estimate

$$\|\Phi_0\|_{W^2_2(R^2_+)} \leqslant \widetilde{C}_6\{\, \|\varphi_1\|_{W^2_2(R^2_+)} + \|\varphi_2\|_{W^2_2(R^2_+)} +, \|p_0^\perp|_{x_1=0}\|_{W^1_2(R^1)} + \|v_{2,0}|_{x_1=0}\|_{W^1_2(R^1)}\,\},$$

where $\widetilde{C}_6 > 0$ is a positive constant, $p_0^\perp = p^\perp|_{t=0}$; $v_{2,0} = v_2|_{t=0}$. Finally, using the property of the trace of a function from $W^1_2(R^2_+)$ along the line $x_1 = 0$, we obtain as a result

$$\|\Phi_0\|_{W^2_2(R^2_+)} \leqslant C_7\{\, \|\varphi_1\|_{W^2_2(R^2_+)} + \|\varphi_2\|_{W^2_2(R^2_+)} + \|p_0^\perp\|_{W^2_2(R^2_+)} + \|v_{2,0}\|_{W^2_2(R^2_+)}\}, \tag{5.28}$$

where $C_7 > 0$ is a positive constant independent of $\varphi_{1,2}$, $p_0^\perp$, $v_{2,0}$.

Considering the function $\Phi$ as auxiliary and taking (5.28) into account, we derive from (5.27) the desired a priori estimate of the solution of the problem $\mathcal{F}$:

$$\|\mathbf{U}(t)\|_{W^2_2(R^2_+)} \leqslant K_1, \qquad 0 < t \leqslant T < \infty. \tag{5.29}$$

Here $K_1 > 0$ is a positive constant determined finally by the value $T$,

$$\|\mathbf{U}(t)\|^2_{W^2_2(R^2_+)} = \iint\limits_{R^2_+} \{(\mathbf{U},\mathbf{U}) + (\mathbf{U}_t,\mathbf{U}_t) + (\mathbf{U}_{x_1},\mathbf{U}_{x_1}) + (\mathbf{U}_{x_2},\mathbf{U}_{x_2})$$
$$+ (\mathbf{U}_{x_1x_1},\mathbf{U}_{x_1x_1}) + (\mathbf{U}_{x_1x_2},\mathbf{U}_{x_1x_2}) + (\mathbf{U}_{x_2x_2},\mathbf{U}_{x_2x_2})\}\, d\mathbf{x}.$$

Adding again (5.24) and (5.9), we obtain

$$\frac{d}{dt} J(t) + \int\limits_{R^1} \mathcal{A}\, dx_2 \leqslant C_5 J(t). \tag{5.30}$$

Integrating (5.30) over the interval $(0, T)$ and taking into account that $J(t) > 0$ and $\mathcal{A} > 0$, with the help of the boundary conditions we find the inequality

$$\int\limits_0^T\int\limits_{R^1} \{(F_t)^2 + (F_{x_2})^2 + (F_{tt})^2 + (F_{tx_2})^2 + \ldots + (F_{x_2x_2x_2})^2\}\, dx_2\, dt \leqslant C_8, \tag{5.31}$$

where $C_8 > 0$ is a positive constant determined finally by the value $T$. From the second and third conditions (2.3), we obtain the equality

$$F_t = \Big(\tilde{\mu} - \frac{N_2\eta}{\lambda_0}\Big)p^\perp + \frac{N_2}{\lambda_0}v_2, \qquad x_1 = 0.$$

Multiplying this equality by $2F$ and integrating it over $x_2 \in R^1$ we obtain, using the Hölder inequality, the estimate

$$\frac{d}{dt}\|F(t)\|^2_{L_2(R^1)} \leqslant c\|F(t)\|_{L_2(R^1)}\{\|p^\perp|_{x_1=0}\|_{L_2(R^1)} + \|v_2|_{x_1=0}\|_{L_2(R^1)}\},$$

where $c > 0$ is a positive constant, $\|F(t)\|^2 = \int\limits_{R^1} F^2 dx_2$, and so on. Using the property of a function trace on the line $x_1 = 0$, the last inequality can be rewritten as

$$\frac{d}{dt}\|F(t)\|_{L_2(R^1)} \leqslant \frac{c\mathrm{M}_b}{2}\left\{\|p^{\perp}(t)\|_{W_2^1(R_+^2)} + \|v_2(t)\|_{W_2^1(R_+^2)}\right\}. \tag{5.32}$$

In (5.32), $\mathrm{M}_b > 0$ is a positive constant. Further, we find from (5.32) by using the proved estimate (5.29)

$$\|F\|_{L_2((0,T)\times R^1)} \leqslant C_9, \tag{5.33}$$

where $C_9 > 0$ is a $T$-dependent positive constant. Then, combining (5.31) and (5.33), we obtain finally the desired a priori estimate for the function $F$

$$\|F\|_{W_2^3((0,T)\times R^1)} \leqslant K_2. \tag{5.34}$$

Here, $K_2 > 0$ is a positive constant determined by $T$.

Thus, the obtained a priori estimates (5.29) and (5.34) prove that the problem $\mathcal{F}$ is well-posed at $q \ll 1$ and, consequently, the fast shock wave is stable in a collisionless magnetized plasma at high pressure.

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